A Note on Set Graceful Labeling of Graphs

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Abstract

We settle affirmatively a conjecture posed in [S. M. Hegde, Set colorings of graphs, *European Journal of Combinatorics* **30** (4) (2009), 986–995]: If some subsets of a set X are assigned injectively to all vertices of a complete bipartite graph G such that the collection of all sets, each of which is the symmetric difference of the sets assigned to the ends of some edge, is the set of all nonempty subsets of X, then G is a star.

Keywords: set graceful labeling of a graph, bipartite graph, symmetric difference of two sets.

2010 Mathematics Subject Classification: 05C78.

Infinite graphs and graphs which are not simple are out of our consideration. Let G be a graph with vertex set V and edge set E and $\mathscr{X}$ be the power set of a set. Let f be a function from V to $\mathscr{X}$; we associate with f, a function from E to $\mathscr{X}$, denoted by $\hat{f}$: for all $xy \in E$, $\hat{f}(xy) = f(x)\Delta f(y)$. (The symmetric difference of two sets A and B is denoted by $A\Delta B$.) If both f and $\hat{f}$ are injective and the range of the latter is $\mathscr{X} \setminus \{\emptyset\}$, then f is called a *set graceful labeling* of G. Settling [1, Conjecture 2] is the objective of this note:

Theorem. *If a complete bipartite graph G has a set graceful labeling, then it is a star.*

Proof. Let V and E be respectively, the vertex set and the edge set of G and P, Q be the bipartition of G. Let $f : V \to \mathscr{X}$ be a set graceful labeling of G. Since $\hat{f}$ is a bijection from E to $\mathscr{X} \setminus \{\emptyset\}$, it follows that $|E| = |\mathscr{X}| - 1$; i.e.,

$$|P||Q| + 1 = |\mathscr{X}|. \tag{1}$$

Suppose that G is not a star. Then $|P| \neq 1 \neq |Q|$; this implies that $(|P|-1)(|Q|-1) > 0$; i.e., $|P||Q| + 1 > |P| + |Q|$; therefore by (1), $|\mathscr{X}| > |V|$; since $f : V \to \mathscr{X}$ is injective, we can find some $A \in \mathscr{X} \setminus \{f(v) : v \in V\}$. Define a map $g : V \to \mathscr{X}$ as follows: for any $v \in V$, $g(v) = A\Delta f(v)$. Since for any $uv \in E$, $g(u)\Delta g(v) =$

$A\Delta f(u)\Delta A\Delta f(v) = f(u)\Delta f(v)$, it follows that $\hat{g} = \hat{f}$; further g is obviously injective; therefore, g is a set graceful labeling. Since $A \notin \{f(v) : v \in V\}$, for each $v \in V$, $g(v) = A\Delta f(v) \neq \emptyset$. Therefore,

$$\emptyset \notin \{g(v) : v \in V\}. \tag{2}$$

Now, let $p \in P$. Let us show that there is exactly one vertex $v \in P$ such that $g(p)\Delta g(v) \in \{g(x) : x \in Q\}$. Since $g(p) \neq \emptyset$ by (2), and the range of $\hat{g}$ is $\mathscr{X} \setminus \{\emptyset\}$, for some $p' \in P$ and $q \in Q$, $g(p) = \hat{g}(p'q)$; i.e., $g(p) = g(p')\Delta g(q)$; therefore, $g(p)\Delta g(p') = g(q)$. Suppose that $p'' \in P$ and $r \in Q$ such that $g(p)\Delta g(p'') = g(r)$. Then $g(p')\Delta g(q) = g(p) = g(p'')\Delta g(r)$; i.e., $\hat{g}(p'q) = \hat{g}(p''r)$. Since $\hat{g}$ is injective, $p' = p''$. Thus for any element $p \in P$, there is a unique element $v \in P$ such that $g(p)\Delta g(v) \in \{g(x) : x \in Q\}$. Define a map $\theta : P \to P$ as follows. For any $v \in P$, let $\theta(v)$ be the (unique) vertex in P such that $g(v)\Delta g(\theta(v)) \in \{g(x) : x \in Q\}$. Note that by (2), for each $v \in P$, $\theta(v) \neq v$ and $\theta^2(v) = v$. Thus $\big\{\{v, \theta(v)\} : v \in P\big\}$ is a partition of P into subsets of order 2. Therefore $|P|$ is even whence by (1), $|\mathscr{X}|$ is odd—a contradiction. ■

References

[1] S. M. Hegde, Set colorings of graphs, *European Journal of Combinatorics* **30** (4) (2009), 986–995.

[2] D. B. West, *Introduction to Graph Theory*, Second edition, Printice Hall, New Jersey, U.S.A. (2001).